Exact solution of the simplest super-orthosymplectic invariant magnet

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Abstract

We present the exact solution of the $Osp(1|2)$ invariant magnet by the Bethe ansatz approach. The associated Bethe ansatz equation exhibit a new feature by presenting an explicit and distinct phase behaviour in even and odd sectors of the theory. The ground state, the low-lying excitations and the critical properties are discussed by exploiting the Bethe ansatz solution.

The concept of integrability based on the Yang-Baxter relation can be extended in order to include in its algebraic structure variables possessing both commuting and anti-commuting rules of permutation. This makes it possible to construct integrable lattice models in which their bond configurations admit bosonic and fermionic degrees of freedom [1]. This notion is then systematized by relating the Boltzmann weights of the lattice model to the R-matrix solutions of a graded version [1] of the Yang-Baxter equation. This generalization is necessary in order to take into account the Grassmann parities coming from the interchange of fermions of the theory. Moreover, for a typical system consisting of n species of bosons and m species of fermions, their solutions are believed to be found as invariants under the $Sl(n|m)$ and $Osp(n|2m)$ superalgebras [2].

The exact solution of the $Sl(n|m)$ invariant systems has been studied in many different contexts in the literature. For instance, the $Sl(n|m)$ property is resembled much on the Perk-Schultz vertex model [3, 4] which possesses an extra (discrete) parameter playing the role of the $Sl(n|m)$ Grasmann parities. More recently, these solutions have also been examined in strongly correlated eletronic systems, being the one dimensional supersymmetric $t - J$ model the typical example [5, 6, 7]. However, for the $Osp(n|2m)$ series, not much is known concerning its Bethe ansatz solutions and critical properties. As far as we know, the only exception is the $U_q Osp(2|2)$ [8] model which has been recently solved in the context of $N = 2$ supersymmetric lattice models [9].

The purpose of this letter is to exactly solve the simplest system in the $Osp(n|2m)$ family, namely the case of one boson ($n = 1$) and two fermions ($m = 1$). As we shall see, a new feature appears in the Bethe ansatz equations, making possible an explicit distinction between the even and odd sectors of the model. This new property has important consequences for the finite-size behaviour which governs the class of universality of this theory.

We begin by introducing the isotropic $Osp(1|2)$ magnet. Its Hamiltonian defined on a lattice of L sites (assuming periodic boundary conditions) can be written in the following

compact form

$$H = -\sum_{i=1}^{L}[P_{i,i+1} + \frac{2}{3}E_{i,i+1}]\tag{1}$$

where $(P_{i,i+1})_{ab}^{cd} = (-1)^{p(a)p(b)}\delta_{a,d}\delta_{b,c}$ is the graded permutation operator and $(E_{i,i+1})_{ab}^{cd} = \alpha_{ab}\alpha_{cd}^{st}$ is the $Osp(1|2)$ Temperely-Lieb invariant operator [10]. The variables $p(a)$ are the Grasmann parities, namely $p(1) = 0$ (boson) and $p(2) = p(3) = 1$ (fermions). The symbol α^{st} denotes the supertranspose operation on a $3X3$ matrix α given by

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\tag{2}$$

The Bethe ansatz solution of (1) is as follows. The first step is to write the corresponding vertex operator of the lattice model leading to the Hamiltonian (1). After an appropriate canonical transformation, it is possible to show that this weight can be expressed in terms of the generators S^z, S^+, S^- of a spin-1 algebra. Its explicit form is

$$R(\lambda) = \begin{pmatrix} \lambda I + f(\lambda)S^z - g(\lambda)(S^z)^2 & \frac{1}{\sqrt{2}}[S^-S^z - 2\lambda h(\lambda)S^zS^-] & f(\lambda)(S^-)^2 \\ \frac{1}{\sqrt{2}}[S^zS^+ - 2\lambda h(\lambda)S^+S^z] & [\lambda - 3h(\lambda)]I + 3h(\lambda)(S^z)^2 & -\frac{1}{\sqrt{2}}[S^zS^- + 2\lambda h(\lambda)S^-S^z] \\ f(\lambda)(S^+)^2 & -\frac{1}{\sqrt{2}}[S^+S^z + 2\lambda h(\lambda)S^zS^+] & \lambda I - f(\lambda)S^z - g(\lambda)(S^z)^2 \end{pmatrix}\tag{3}$$

where I is the $3X3$ identity matrix and functions $f(\lambda)$, $g(\lambda)$ and $h(\lambda)$ are given by

$$f(\lambda) = \frac{(2\lambda - 3/2)}{2(\lambda - 3/2)}; \quad h(\lambda) = \frac{1}{2(\lambda - 3/2)}; \quad g(\lambda) = 2\lambda + \frac{3}{2}h(\lambda)\tag{4}$$

It is important to notice that the reference state preserving the triangularity property of this R-matrix is still the usual ferromagnetic pseudo-vaccum. Moreover, we recall that its structure resembles much the one that appears in the Izergin-Korepin (IK) model [11, 12]. Indeed we are able to show that the Hamiltonian (1) can be obtained as a certain branch limit of the IK system through a delicate but rigorous canonical transformation. The calculations leading to this result are rather cumbersome and will be presented elsewhere [13]. Here we can use this information ,by adapting the algebraic and(or) the analytic

Bethe ansatz method used in the IK model [12], in order to solve Hamiltonian (1). Due to a $U(1)$-conserved charge the $Osp(1|2)$ Hamiltonian can be block diagonalized into disjoint sectors labelled by the total magnetization index, namely $r = \sum_{i=1}^{L} S_i^z$. In a given sector r, for periodic boundary condition, the eigenstates are parametrized by a set of complex numbers λ_j satisfying the following Bethe ansatz equation

$$\left(\frac{\lambda_j - i/2}{\lambda_j + i/2}\right)^L = -(-1)^r \prod_{k=1}^{L-r} \left(\frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k + i}\right) \left(\frac{\lambda_j - \lambda_k + i/2}{\lambda_j - \lambda_k - i/2}\right) \tag{5}$$

and the eigenenergies are given by

$$E_r(L) = -\sum_{j=1}^{L-r} \frac{1}{\lambda_j^2 + 1/4} + L \tag{6}$$

The important novelty present in the Bethe ansatz equation is its dependence on a phase factor $(-1)^r$ even for a system with periodic boundary conditions. Hence, even and odd sectors of the theory are distinguished by the signs of their "bare" phase-shift (right hand side of (4)). This may be interpreted as an explicit separation of the bosonic and fermionic degrees of freedom present in the system. In fact, we have verified this "sector separation" through an exact diagonalization of the Hamiltonian (1) for small values of L. We also remark that although a phase factor does not affect the basic properties in the thermodynamic limit $(L \to \infty)$, they do change the finite-size behaviour. Therefore, this phase factor plays an important role in order to characterize the correct critical behaviour of the model.

Before turning to the study of the finite size properties of (4) and (5), we first compute some useful quantities in the thermodynamic limit. The ground state in each sector r is parametrized by a set of real numbers λ_j. Taking the logarithm of equation (4) and correctly collecting the phase factors, we rewrite the Bethe ansatz equations as

$$L\psi_{1/2}(\lambda_j) = 2\pi Q_j + \sum_{k=1,k\neq j}^{L-r} [\psi_1(\lambda_j - \lambda_k) - \psi_{1/2}(\lambda_j - \lambda_k)] \tag{7}$$

where $\psi_a(x) = 2\arctan(x/a)$ and Q_j are integer or semi-integer series of number

$$Q_j = -\frac{[L-r-1]}{2} + j - 1, \quad j = 1, 2, \cdots, L-r \tag{8}$$

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For large L, the roots λ_j are densely packed into its density distribution, $\sigma_L(\lambda_j) \cong 1/L(\lambda_{j+1} - \lambda_j)$. Strictly in the $L \to \infty$ limit this system can be solved by elementary Fourier techniques and we find that

$$\sigma(\lambda) = \frac{2}{\sqrt{3}} \frac{\cosh(2\pi\lambda/3)}{\cosh(4\pi\lambda/3) + 1/2} \tag{9}$$

The ground state energy (per site) can be calculated by using equations (6) and (9) after replacing the sum by an integral. The final result is

$$e_\infty = \lim_{L \to \infty} E_0(L)/L = -\frac{4\pi\sqrt{3}}{9} + 1 \cong -1.4184 \tag{10}$$

In addition, we have verified that the low-lying excitations over the ground state are gapless. As usual, the excitations are built up by inserting vacancies on the density distribution of λ_j. The simplest excitation is the triplet state made of two real "holes" λ_1 and λ_2. Their contribution to the total momenta is calculated to be

$$p(\lambda_1, \lambda_2) = \pi - \sum_{i=1}^{2} 2 \arctan[\sinh(2\pi\lambda_i/3)/\cosh(\pi/6)] \tag{11}$$

which leads to the following low-momentum dispersion relation

$$\epsilon(p) \sim v_s p \tag{12}$$

where $v_s = 2\pi/3$ is the sound velocity.

At this point we have obtained the most important quantities necessary for the analysis of the critical properties. The class of universality can be determined by exploiting a set of important relations between the eigenspectrum of finite-lattice system [14]. For instance, the central charge c is related to the ground state energies $E_0(L)$ by [15]

$$E_0(L)/L - e_\infty \cong -\frac{\pi c v_s}{6L^2} \tag{13}$$

In Table 1 we present our results for the central charge c. This result is obtained by numerically solving the Bethe ansatz equation in the sector $r = 0$. The extrapoled results predict a conformal anomaly $c = 1$. The sectors $r \neq 0$ are responsible for the low-lying

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conformal dimensions. In this case , however, a numerical study up to lattice size $L \sim 64$ presents a poor convergence rate. This is due to the appearance of logarithm corrections of order $O(1/\ln(L))$ in the scaling function. Fortunately, since the roots λ_j are real, we can use an analytical method developed by De Vega and Woynarovich [16] in order to estimate these critical dimensions. The calculation is quite technical [13] and here we only present the final results. We stress, however, that the phase factor of (5) is of fundamental importance in this computation. The critical dimension X_r associated with the lowest operator on sector r has the following form

$$X_r = \frac{r^2}{4} \tag{14}$$

Such dimension corresponds to a spin-wave excitation of index r in a Gaussian structure of operators. In fact, more generally we expect to find the following dimensions

$$X_{r,m} = \frac{r^2}{4} + m^2 \tag{15}$$

where m stands for the "vortex" excitations index. On the other hand, this result can be justified by using the following arguments of symmetry. For example, we have verifed that (for finite lattices) the first excited state with zero momentum in the $r = 0$ sector is degenerated to the ground state in sector $r = 2$. This means the identity $X_{0,1} = X_{2,0}$, which is indeed verified in formula (15). Evidently, the same argument can be repeated for many other values of r and m, confirming the conjecture (15). We recall that this is the manifestation of the $Osp(1|2)$ symmetry encoded in the important phase factor $(-1)^r$.

In summary, we have given a Bethe ansatz solution to the eigenvalues of the $Osp(1|2)$ magnet. The novelty is the presence of a phase factor depending on the sector r, which becomes crucial in order to determine its critical properties. We believe that our results extends the form of Bethe ansatz solution appearing in integrable lattice models. Hopefully, our results will be of relevance in the discussion of the Bethe ansatz completeness and in the thermodynamic properties.

Acknowledgements

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Table Captions

Table 1. The estimative of the conformal anomaly of equation (13)

Table 1

L	$-6L^2[E_0(L)/L - e_\infty]/\pi v_s$
16	1.0029 729
24	1.0022 324
32	1.0018 406
40	1.0015 955
48	1.0014 261
56	1.0013 010
64	1.0012 043
Extrapolated	1.0001 (1)